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A Remark on Simultaneous Approximation

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In this paper we generalize a theorem of [1], and prove a theorem on the continuity of the simultaneous approximation operator. The notation and definitions are adapted from [1, 2].

THEOREM 1. *Let $(X, \|\cdot\|)$ be a strictly convex normed linear space, and let M be a reflexive subspace of $(X, \|\cdot\|)$. Then for every nonempty compact (in the norm-topology) set $F \subset X$ there exists a unique simultaneous best approximation point in M.*

Proof. Let the mapping $\Phi_F : M \to R$ be defined by

$$\Phi_F(m) = \sup_{f\in F} \| f - m \|.$$

Since the norm-function is weakly lower semicontinuous, the function Φ_F is weakly lower semicontinuous.

On the other hand,

$$\inf_{m\in M} \Phi_F(m) \equiv \varphi_M(F) \leqslant \Phi_F(0) = \sup_{f\in F} \|f\|,$$

which implies

$$\varphi_M(F) < \Phi_F(m'),$$

if

$$\| m' \| > 2 \sup_{f\in F} \|f\|.$$

From this,

$$\varphi_M(F) = \inf_{m\in A} \Phi_F(m),$$

where

$$A = U(0, 2 \sup_{f\in F} \|f\|) \cap M.$$

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By the reflexivity of M, the set A is weakly compact. There exists $m_0 \in A$ such that

$$\varphi_M(F) = \Phi_F(m_0).$$

Now, we prove that Φ_F attains its minimum in M at exactly one point.

Assuming the contrary, there is a point $m_0' \in A$ ($m_0' \neq m_0$) such that

$$\Phi_F(m_0) = \Phi_F(m_0').$$

Then

$$\inf_{m \in M'} \Phi_F(m) = \Phi_F(m_0) = \Phi_F(m_0'),$$

where M' denotes the linear hull of m_0 and m_0'. From this, using [1, Theorem 1], we have $m_0 = m_0'$. ∎

Let us denote by cpX the metric space of all nonempty compact sets $F \subset X$ with the Hausdorff metric

$$d(G_1, G_2) = \max\{\sup_{g_1 \in G_1} \inf_{g_2 \in G_2} \| g_1 - g_2 \|, \sup_{g_2 \in G_2} \inf_{g_1 \in G_1} \| g_1 - g_2 \|\}.$$

Let $P_M : cpX \to M$ denote the simultaneous best approximation operator, if it exists and is single valued.

THEOREM 2. *Let $(X, \| \cdot \|)$ be a reflexive, locally uniformly convex Banach space, and let $M \subset X$ be a closed subspace. Then the mapping $P_M : cpX \to M$ is continuous.*

Proof. The existence of P_M is obvious by Theorem 1. Assume that $F_n, F \in cpX$, $d(F_n, F) \to 0$. Then

$$\bigcap_{n=1}^{\infty} \left[M \cap \bigcap_{x \in F} U(x, \varphi_M(F) + 2d(F_n, F)) \right] = P_M(F).$$

The relation

$$P_M(F_n) \in M \cap \bigcap_{x \in F} U(x, \varphi_M(F) + 2d(F_n, F))$$

implies the existence of a subsequence $\{P_M(F_{n_K})\}_{K=1}^{\infty}$ of the sequence $\{P_M(F_n)\}_{n=1}^{\infty}$ for which

$$\exists(w) \lim_K P_M(F_{n_K}) = P_M(F). \tag{1}$$

By compactness of F, there exists $x_0 \in F$ such that

$$\varphi_M(F) = \| x_0 - P_M(F) \|.$$

It is obvious that for any $G_1, G_2 \in cpX$,

$$| \varphi_M(G_1) - \varphi_M(G_2)| \leqslant d(G_1, G_2). \tag{2}$$

From $d(F_n, F) \to 0$ it follows that $\exists \{x_{n_{K_\gamma}}\}_{\gamma=1}^{\infty}$ such that $x_{n_{K_\gamma}} \in F_{n_{K_\gamma}}$, $\exists \lim_\gamma x_{n_{K_\gamma}} = x_0$, and $\exists \lim_\gamma \| x_{n_{K_\gamma}} - P_M(F_{n_{K_\gamma}})\|$.

Using (2),

$$\lim_\gamma \| x_{n_{K_\gamma}} - P_M(F_{n_{K_\gamma}})\| \leqslant \| x_0 - P_M(F)\|.$$

Moreover, we shall prove

$$\lim_\gamma \| x_{n_{K_\gamma}} - P_M(F_{n_{K_\gamma}})\| = \| x_0 - P_M(F)\|. \tag{3}$$

For the proof we need relation (4) which follows directly from (1):

$$(w) \lim_\gamma (x_{n_{K_\gamma}} - P_M(F_{n_{K_\gamma}})) = x_0 - P_M(F). \tag{4}$$

Were (3) false, there would be an $\epsilon > 0$ and $\gamma_0 \in \mathbb{N}$ such that $\forall \gamma \geqslant \gamma_0$,

$$x_{n_{K_\gamma}} - P_M(F_{n_{K_\gamma}}) \in U(0, \| x_0 - P_M(F)\| - \epsilon).$$

From the convexity and closure of the unit ball, $U(0, \| x_0 - P_M(F)\| - \epsilon)$ is weakly closed. So

$$(w) \lim_\gamma (x_{n_{K_\gamma}} - P_M(F_{n_{K_\gamma}})) \in U(0, \| x_0 - P_M(F)\| - \epsilon),$$

contradicting (4).

The local uniform convexity of the norm implies [2, p. 32, Theorem 4]

$$\lim_\gamma (x_{n_{K_\gamma}} - P_M(F_{n_{K_\gamma}})) = x_0 - P_M(F),$$

which implies, in turn,

$$\lim_\gamma P_M(F_{n_{K_\gamma}}) = P_M(F).$$

Theorem 2 now follows by standard arguments.

References

1. A. S. Holland, B. N. Sahney, and J. Tzimbalario, On best simultaneous approximation, Letter to the Editor, *J. Approximation Theory* **17** (1976), 187–188.
2. J. Diestel, "Geometry of Banach Spaces—Selected Topics," Lecture Notes in Mathematics, No. 485, Springer-Verlag, Berlin/Heidelberg/New York, 1975.

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